Baltic Way 2017

Version: *English*

Sorø, November 11th, 2017

Time allowed: 4.5 hours.
During the first 30 minutes, questions may be asked.
Tools for writing and drawing are the only ones allowed.

Problem 1. Let $a_0, a_1, a_2, \ldots$ be an infinite sequence of real numbers satisfying $\frac{a_{n-1}+a_{n+1}}{2} \geq a_n$ for all positive integers n. Show that

$$\frac{a_0 + a_{n+1}}{2} \geq \frac{a_1 + a_2 + \ldots + a_n}{n}$$

holds for all positive integers n.

Problem 2. Does there exist a finite set of real numbers such that their sum equals 2, the sum of their squares equals 3, the sum of their cubes equals 4, ..., and the sum of their ninth powers equals 10?

Problem 3. Positive integers $x_1, \ldots, x_m$ (not necessarily distinct) are written on a blackboard. It is known that each of the numbers $F_1, \ldots, F_{2018}$ can be represented as a sum of one or more of the numbers on the blackboard. What is the smallest possible value of m?

(Here $F_1, \ldots, F_{2018}$ are the first 2018 Fibonacci numbers: $F_1 = F_2 = 1$, $F_{k+1} = F_k + F_{k-1}$ for $k > 1$.)

Problem 4. A *linear form* in k variables is an expression of the form $P(x_1, \ldots, x_k) = a_1 x_1 + \ldots + a_k x_k$ with real constants $a_1, \ldots, a_k$. Prove that there exist a positive integer n and linear forms $P_1, \ldots, P_n$ in 2017 variables such that the equation

$$x_1 \cdot x_2 \cdot \ldots \cdot x_{2017} = P_1(x_1, \ldots, x_{2017})^{2017} + \ldots + P_n(x_1, \ldots, x_{2017})^{2017}$$

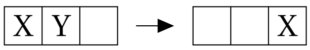
holds for all real numbers $x_1, \ldots, x_{2017}$.

Problem 5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 y) = f(xy) + y f(f(x) + y)$$

for all real numbers x and y.

Problem 6. Fifteen stones are placed on a 4×4 board, one in each cell, the remaining cell being empty. Whenever two stones are on neighbouring cells (having a common side), one may jump over the other to the opposite neighbouring cell, provided this cell is empty. The stone jumped over is removed from the board.

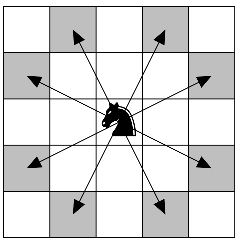


For which initial positions of the empty cell is it possible to end up with exactly one stone on the board?

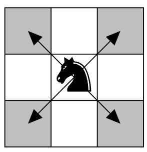
Problem 7. Each edge of a complete graph on 30 vertices is coloured either red or blue. It is allowed to choose a non-monochromatic triangle and change the colour of the two edges of the same colour to make the triangle monochromatic. Prove that by using this operation repeatedly it is possible to make the entire graph monochromatic.

(A complete graph is a graph where any two vertices are connected by an edge.)

Problem 8. A chess knight has injured his leg and is limping. He alternates between a normal move and a short move where he moves to any diagonally neighbouring cell.



Normal move



Short move

The limping knight moves on a 5×6 cell chessboard starting with a normal move. What is the largest number of moves he can make if he is starting from a cell of his own choice and is not allowed to visit any cell (including the initial cell) more than once?

Problem 9. A positive integer n is *Danish* if a regular hexagon can be partitioned into n congruent polygons. Prove that there are infinitely many positive integers n such that both n and $2^n + n$ are Danish.

Problem 10. Maker and Breaker are building a wall. Maker has a supply of green cubical building blocks, and Breaker has a supply of red ones, all of the same size. On the ground, a row of m squares has been marked in chalk as place-holders. Maker and Breaker now take turns in placing a block either directly on one of these squares, or on top of another block already in place, in such a way that the height of each column never exceeds n. Maker places the first block.

Maker bets that he can form a green row, i.e. all m blocks at a certain height are green. Breaker bets that he can prevent Maker from achieving this. Determine all pairs (m, n) of positive integers for which Maker can make sure he wins the bet.

Problem 11. Let H and I be the orthocentre and incentre, respectively, of an acute angled triangle ABC. The circumcircle of the triangle BCI intersects the segment AB at the point P different from B. Let K be the projection of H onto AI and Q the reflection of P in K. Show that B, H and Q are collinear.

Problem 12. Line ℓ touches circle S_1 in the point X and circle S_2 in the point Y. We draw a line m which is parallel to ℓ and intersects S_1 in a point P and S_2 in a point Q. Prove that the ratio XP/YQ does not depend on the choice of m.

Problem 13. Let ABC be a triangle in which $\angle ABC = 60°$. Let I and O be the incentre and circumcentre of ABC, respectively. Let M be the midpoint of the arc BC of the circumcircle of ABC, which does not contain the point A. Determine $\angle BAC$ given that $MB = OI$.

Problem 14. Let P be a point inside the acute angle $\angle BAC$. Suppose that $\angle ABP = \angle ACP = 90°$. The points D and E are on the segments BA and CA, respectively, such that $BD = BP$ and $CP = CE$. The points F and G are on the segments AC and AB, respectively, such that DF is perpendicular to AB and EG is perpendicular to AC. Show that $PF = PG$.

Problem 15. Let $n \geq 3$ be an integer. What is the largest possible number of interior angles greater than $180°$ in an n-gon in the plane, given that the n-gon does not intersect itself and all its sides have the same length?

Problem 16. Is it possible for any group of people to choose a positive integer N and assign a positive integer to each person in the group such that the product of two persons' numbers is divisible by N if and only if they are friends?

Problem 17. Determine whether the equation

$$x^4 + y^3 = z! + 7$$

has an infinite number of solutions in positive integers.

Problem 18. Let $p > 3$ be a prime and let $a_1, a_2, \ldots, a_{\frac{p-1}{2}}$ be a permutation of $1, 2, \ldots, \frac{p-1}{2}$. For which p is it always possible to determine the sequence $a_1, a_2, \ldots, a_{\frac{p-1}{2}}$ if for all $i, j \in \{1, 2, \ldots, \frac{p-1}{2}\}$ with $i \neq j$ the residue of $a_i a_j$ modulo p is known?

Problem 19. For an integer $n \geq 1$ let $a(n)$ denote the total number of carries which arise when adding 2017 and $n \cdot 2017$. The first few values are given by $a(1) = 1$, $a(2) = 1$, $a(3) = 0$, which can be seen from the following:

001	001	000
2017	4034	6051
+2017	+2017	+2017
=4034	=6051	=8068

Prove that

$$a(1) + a(2) + \ldots + a(10^{2017} - 2) + a(10^{2017} - 1) = 10 \cdot \frac{10^{2017} - 1}{9}.$$

Problem 20. Let S be the set of all ordered pairs (a, b) of integers with $0 < 2a < 2b < 2017$ such that $a^2 + b^2$ is a multiple of 2017. Prove that

$$\sum_{(a,b)\in S} a = \frac{1}{2} \sum_{(a,b)\in S} b.$$